On the geometry of impulsive gravitational waves

R. Steinbauer [*]

Department of Mathematics, University of Vienna, Strudlhofg. 4
A-1090 Wien, Austria
and
Institute for Theoretical Physics, University of Vienna, Boltzmanng. 5
A-1090 Wien, Austria

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Abstract

We describe impulsive gravitational pp-waves entirely in the distributional picture. Applying Colombeau's nonlinear framework of generalized functions we handle the formally ill-defined products of distributions which enter the geodesic as well as the geodesic deviation equation. Using a universal regularization procedure we explicitly derive regularization independent distributional limits. In the special case of impulsive plane waves we compare our results with the particle motion derived from the continuous form of the metric.

[*]Electronic mail: roland.steinbauer@univie.ac.at

1 Introduction

Plane fronted gravitational waves with parallel rays (pp-waves) are spacetimes admitting a covariantly constant null vector field, which can be used to write the metric tensor in the form [1]

$$ds^2 \ = \ H(u,x,y)du^2 - du\,dv + dx^2 + dy^2, \tag{1}$$

where u, v and x, y is a pair of null and transverse (Cartesian) coordinates respectively. In this work we shall deal especially with *impulsive* pp-waves which can be described by a profile function H proportional to a δ-distribution [2], i.e. $H(u,x,y) = f(x,y)\delta(u)$, where f is assumed to be smooth. Hence the spacetime is flat everywhere, exept for the null hypersurface $u = 0$, where it has a δ-shaped "shock". Such geometries arise physically as ultrarelativistic limits of boosted black hole spacetimes of the Kerr-Newman family [3, 4] and multipole solutions of the Weyl family [5]. Moreover they play an important role in particle scattering at the Planck scale [6].

There are also intrinsic descriptions of impulsive pp-waves, most prominently, Penrose's "scissors and paste approach" [2], which consists in glueing together two copies of Minkowski space along the null hypersurface $u = 0$, identifying points according to $(0, v, x, y) = (0, v + (1/2)f(x,y), x, y)$. Penrose also introduced a different coordinate system in which the components of the metric tensor are actually continuous and, in the special case of a *plane* wave, i.e. $f(x,y) = x^2 - y^2$ take the form

$$ds^2 \ = \ -du\,dV + (1 - u_+)^2 dX^2 + (1 + u_+)^2 dY^2, \tag{2}$$

where $u_+ = \theta(u)u$ denotes the "kink-function" and θ is the step function. Clearly, the transformation relating (1) with (2) has to be discontinuous, i.e. is given by [2] (for the case of a general pp-wave see [7])

$$
\begin{aligned}
x &= (1 + u_+)X \\
y &= (1 - u_+)Y \\
v &= V + X^2(u+1)\theta(u) + Y^2(u-1)\theta(u)
\end{aligned}
\tag{3}
$$

Hence –strictly speaking– the (topological) structure of the manifold is changed. In this work we begin our investigations using the original distributional form of the metric, motivated by the fact that physically, i.e. in the ultrarelativistic limit, the spacetime arises that way (cf. also the approaches of [8, 9]). Solving the geodesic as well as the geodesic deviation equation, we describe the geometry of impulsive pp-waves entirely in the distributional picture. As discussed in detail in [10] these equations, due to their nonlinearity and the presence of the Dirac δ-function in the spacetime metric, involve formally ill-defined products of distributions. To overcome this problem we use the setting of Colombeau algebras of generalized functions [11, 12, 13], which we briefly describe in Sec. 2.

Finally, for the special case of plane impulsive waves, we compare our results with calculations using the continuous coordinate system (see also [14]) to find that the particle motion in both spacetimes agree. Hence we argue that from a physical point of view the spacetime manifolds are identical.

2   Colombeau Algebras

In the vector space of distributions $\mathcal{D}'$ no meaningful product can be defined. Moreover, as *L. Schwartz* [15] showed in 1954, there does not even exist an associative and commutative differential algebra containing the space of C^k-*functions* as a subalgebra that allows a linear embedding of distributions. *J.F. Colombeau* [11, 12, 13] introduced differential algebras $\mathcal{G}$ containing the space of distributions as a subspace, and the space of *smooth* functions as a faithful subalgebra. In the light of Schwartz's so called "impossibility result" this framework has to be regarded as the best one can hope for.

To begin with, we give a short description of the algebra we are going to use in the sequel which provides a natural framework for studying nonlinear operations on singular data, hence singular (linear and nonlinear) partial differential equations. The main idea is to regularize singular objects by sequences of smooth functions, i.e. in the space $\mathcal{E}(\Omega) := C^\infty(\Omega)^{(0,1)}$, where Ω denotes an arbitrary open set of R^n. To achieve the consistency properties discussed above we introduce the spaces

$$
\begin{aligned}
\mathcal{E}_\mathcal{M}(\Omega) \;=\; & \{\, (u_\epsilon)_\epsilon \in (C^\infty(\Omega))^{(0,1)} : \forall K \subset\subset \Omega \,\forall \alpha \in \mathbb{N}_0^n \,\exists N > 0 : \\
& \sup_{x \in K} \mid \partial^\alpha u_\epsilon(x) \mid = O(\epsilon^{-N}) \quad (\epsilon \to 0)\,\} \,,
\end{aligned}
\tag{4}
$$

$$
\begin{aligned}
\mathcal{N}(\Omega) \;=\; & \{\, (u_\epsilon)_\epsilon \in (C^\infty(\Omega))^{(0,1)} : \forall K \subset\subset \Omega \,\forall \alpha \in \mathbb{N}_0^n, \forall M > 0 : \\
& \sup_{x \in K} \mid \partial^\alpha u_\epsilon(x) \mid = O(\epsilon^{M}) \quad (\epsilon \to 0)\,\} \,.
\end{aligned}
\tag{5}
$$

$\mathcal{E}_\mathcal{M}(\Omega)$ is a differential algebra with pointwise operations and $\mathcal{N}(\Omega)$ is an ideal in it. We define the *algebra of generalized functions,* or *Colombeau algebra,* by the quotient

$$
\mathcal{G}(\Omega) \;:=\; \mathcal{E}_\mathcal{M}(\Omega)\,/\,\mathcal{N}(\Omega)
\tag{6}
$$

and denote its elements by $u = (u_\epsilon)_\epsilon + \mathcal{N}(\Omega)$ or-more carelessly-by u_ϵ.

Distributions with compact support are now embedded into $\mathcal{G}(\mathbb{R}^n)$ by convolution with a *mollifier* ρ_ϵ, defined as follows; let $\rho \in \mathcal{S}(\mathbb{R}^n)$ (Schwartz's space) with the properties $\int \rho(x)\,dx = 1$ and $\int x^\alpha \rho(x)\,dx = 0 \quad \forall \alpha \in \mathbb{N}^n, \mid \alpha \mid \geq 1$, then we set $\rho_\epsilon(x) := (1/\epsilon^n)\rho(x/\epsilon)$. So we have the map $\iota_0(\omega) = (\omega * \rho_\epsilon)_\epsilon + \mathcal{N}(\mathbb{R}^n)$. This embedding can be "lifted" to an embedding $\iota : \mathcal{D}'(\Omega) \hookrightarrow \mathcal{G}(\Omega)$ by means of sheaf theory while smooth functions are embedded as constant sequences, i.e. $\sigma(f) = (f)_\epsilon$.

Next we briefly recall the concept of association in the Colombeau algebra, which is essential in applications. Two generalized functions u, v are called *associated*

with each other ($u \approx v$), provided their difference converges to 0 in $\mathcal{D}'$. In particular, if $u \in \mathcal{D}'$ then it is called the macroscopic aspect (or *distributional shadow*) of v. Equality in $\mathcal{D}'$ is reflected as equality in the sense of association in $\mathcal{G}$, while equality in $\mathcal{G}$ is a stricter concept (for example, all powers of the Heaviside function are distinct in the Colombeau algebra although they are associated with each other; moreover not every Colombeau function "casts a distributional shaddow," for example $\iota(\delta)^2$).

The construction introduced above is known as the "special" or "simplified" variant of Colombeau's algebra. A more refined construction, i.e. the "full" algebra, avoids the dependence on ρ of the embedding of $\mathcal{D}'$. At the expense of some technicalilties one can also construct a diffeomorphism invariant embedding. However, for the present application it will be sufficient to work in the "special" algebra.

In the mathematical literatur Colombeau algebras have been extensively used to study nonlinear PDEs with singular data or coefficients (see eg. [13] and the references therein). Recently there have also been a number of applications to general relativity, in particular the calculation of the curvature of cosmic strings [16, 17, 18] and the ultrarelativistic limits of Kerr-Newman black holes (see [19, 20] and references therein). For a current overview of the topic see the review article of J. Vickers [21].

3 Solving the Geodesic and Geodesic Deviation Equation in $\mathcal{G}$

We start with a pp-metric of the form

$$ds^2 \;=\; f(x^i)\,\delta(u)\,du^2 - du\,dv + (dx^i)^2 \;, \tag{7}$$

where x^i ($i = 1, 2$) denote the transverse coordinates. It is straightforward to derive the geodesic equation which (using u as an affine parameter, thereby excluding only trivial geodesics parallel to the shock) takes the form

$$
\begin{aligned}
\ddot{v}(u) &= f(x^j(u))\,\dot{\delta}(u) \,+\, 2\,\partial_i\,f(x^j(u))\,\dot{x}^i(u)\,\delta(u) \;, \\
\ddot{x}^i(u) &= \frac{1}{2}\,\partial_i\,f(x^j(u))\,\delta(u) \;,
\end{aligned}
\tag{8}
$$

where $\dot{}$ denotes the derivative with respect to u.

Note that the first line involves a product of the δ-function with $\dot{x}^i$ which we cannot even expect to be continuous (and in fact turns out to be proportional to the Heaviside function, cf. [10]). We proceed by transferring the geodesic equations into Colombeau's framework. The general strategy for solving differential equations in $\mathcal{G}$ is to embed singularities (in our case: δ) into $\mathcal{G}$, which amounts to a regularization, and then solve the corresponding regularized equations. In

order to obtain general results we are therefore interested in imposing as few restrictions as possible on the regularization of δ. The largest "reasonable" class of smooth[1] regularizations of δ is given by nets $(\rho_\varepsilon)_{\varepsilon \in (0,1)}$ of smooth functions ρ_ε satisfying:

$$\text{(a)} \qquad \text{supp}(\rho_\varepsilon) \to \{0\} \quad (\varepsilon \to 0) \ ,$$

$$\text{(b)} \qquad \int \rho_\varepsilon(x)\, dx \to 1 \quad (\varepsilon \to 0) \ \text{ and}$$

$$\text{(c)} \qquad \int |\rho_\varepsilon(x)|\, dx \ \text{is bounded uniformly for small } \epsilon$$

(cf. the definition of *strict delta nets* in [13], ch. 2.7). Obviously any such net converges to δ in distributions as $\varepsilon \to 0$. To simplify notations it is often convenient to replace (a) by

$$\text{(a')} \quad \text{supp}(\rho_\varepsilon) \subseteq [-\varepsilon, \varepsilon] \quad \forall \varepsilon \in (0,1).$$

We shall call a net satisfying conditions (a'), (b) and (c) a *generalized delta function*. Denoting the $\mathcal{G}$-functions corresponding to x^i and v by x_ϵ^i and v_ϵ we are now prepared to state the following (cf. [22])

Theorem 1 *Let $\rho_\epsilon \in \mathcal{G}(\mathbb{R})$ be a generalized delta function. The regularized geodesic equation*

$$\begin{aligned}
\ddot{v}_\epsilon(u) &= f(x_\epsilon^j(u))\, \dot{\rho}_\epsilon(u) + 2\, \partial_i\, f(x_\epsilon^j(u))\, \dot{x}_\epsilon^i(u)\, \rho_\epsilon(u) \ , \\
\ddot{x}_\epsilon^i(u) &= \tfrac{1}{2}\, \partial_i\, f(x_\epsilon^j(u))\, \rho_\epsilon(u)
\end{aligned} \qquad (9)$$

with initial conditions[2] $v_\epsilon(-1) = v_0$, $\dot{v}_\epsilon(-1) = \dot{v}_0$, $x_\epsilon^i(-1) = x_0^i$, $\dot{x}_\epsilon^i(-1) = \dot{x}_0^i$ has a unique locally bounded solution $(v_\epsilon, x_\epsilon^i) \in \mathcal{G}(\mathbb{R})^3$. Moreover these solutions satisfy the following association relations

$$\begin{aligned}
x_\epsilon^i &\approx x_0^i + \dot{x}_0^i(1+u) + \tfrac{1}{2}\partial_i f(x_0^i + \dot{x}_0^i)u_+ \\
v_\epsilon &\approx v_0 + \dot{v}_0(1+u) + f(x_0^i + \dot{x}_0^i)\theta(u) + \\
&\quad + \partial_i f(x_0^i + \dot{x}_0^i)\left(\dot{x}_0^i + \tfrac{1}{4}\partial_i f(x_0^i + \dot{x}_0^i)\right)u_+ \ .
\end{aligned} \qquad (10)$$

Hence from the distributional point of view the geodesics are given by refracted, broken straight lines. More precisely, the geodesics suffer a jump and a kink in the v- as well as a kink in the x^i-direction when crossing the shock hypersurface.

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[1] Note that, since $\mathcal{D}$ is dense in L^1, practically even discontinuous regularizations (eg. boxes) are included.

[2] Note that we have to impose initial conditions "long before" the shock. Geodesics starting at the shock cannot be treated in a regularization independent manner.

The scale of the effect is entirely determined by the values of the profile function and its first derivatives at the shock hypersurface and (in the special case of plane waves) reproduces exactly Penrose's junction conditions (3). Note, however, that the above results are regularization independent even within the maximal class of regularizations of the Dirac δ.

Our next goal is an analysis of the geodesic deviation equation for impulsive pp-waves in the framework of algebras of generalized functions. As in [10] to keep formulas more transparent we make some simplifying assumptions concerning geometry (namely axisymmetry) and initial conditions. Writing $x = x^1$ and $y = x^2$ we suppose that f depends exclusively on the two-radius $\sqrt{x^2 + y^2}$ and work within the hypersurface $y = 0$ (corresponding to initial conditions $y_0 = 0 = \dot{y}_0$). Furthermore we demand $v_0 = 0 = \dot{x}_0$. In this situation the Jacobi equation for the regularized deviation vector field $N_\epsilon^a(u) = (N_\epsilon^u(u), N_\epsilon^v(u), N_\epsilon^x(u), N_\epsilon^y(u))$ takes the form

$$\ddot{N}_\epsilon^u = 0 \qquad \ddot{N}_\epsilon^y = 0 \qquad \dot{N}_\epsilon^x = [\dot{N}_\epsilon^u f'(x_\epsilon) + \frac{1}{2} N_\epsilon^x f''(x_\epsilon)]\rho_\epsilon + \frac{1}{2} f'(x_\epsilon)N_\epsilon^u \dot{\rho}_\epsilon$$

$$\ddot{N}_\epsilon^v = 2[N_\epsilon^x f'(x_\epsilon)\rho_\epsilon]^\cdot - N_\epsilon^x f'(x_\epsilon)\dot{\rho}_\epsilon + [N_\epsilon^u f(x_\epsilon)\rho_\epsilon]^{\cdot\cdot} \qquad (11)$$
$$-N_\epsilon^u f''(x_\epsilon)\dot{x}_\epsilon^2\rho_\epsilon - N_\epsilon^u f'(x_\epsilon)\ddot{x}_\epsilon\rho_\epsilon \,,$$

where $'$ denotes derivatives with respect to r and we have suppressed the parameter u. Furthermore x_ϵ is determined by (9) with simplifiactions as discussed above. Note that the last equation in (11) involves terms proprotional to $\theta^2\delta$ and even δ^2. However, existence and uniqueness of solutions to the corresponding initial value problem in the Colombeau algebra as well as (regularization independent!) association relations (which we give explicitly for some special choices of initial conditions) are still guaranteed.

Theorem 2 *The Jacobi equation (11) with initial conditions $N_\epsilon^a(-1) = n^a$ and $\dot{N}_\epsilon^a(-1) = \dot{n}^a$ has a unique solution in $\mathcal{G}(\mathbb{R})^4$. If $N_\epsilon^a(-1) = 0$ and $\dot{N}_\epsilon^a(-1) = (a, b, 0, 0)$ the unique solution satisfies the following association relations*

$$N_\epsilon^x \approx \frac{1}{2}af'(x_0)(u_+ + \theta(u))$$

$$N_\epsilon^v \approx b(1 + u) + a[f(x_0)\delta(u) + \frac{1}{4}f'(x_0)^2(\theta(u) + u_+)]\,. \qquad (12)$$

For initial conditions $N_\epsilon^a(-1) = (0, 0, a, 0)$, $\dot{N}_\epsilon^a(-1) = 0$ we have

$$N_\epsilon^x \approx a(1 + u_+)$$

$$N_\epsilon^v \approx af'(x_0)(\theta + \frac{1}{2}f''(x_0)u_+)\,. \qquad (13)$$

Hence, viewed distributionally, in the first case the Jacobi field suffers a kink, a jump and a δ-like pulse in the v-direction as well as a kink and jump in

the x-direction overlapping the linear flat space behavior. These effects can be understood heuristically from the corresponding behavior of the geodesics, given by equation (11). The constant factor a, which gives the "scale" of all the nonlinear effects, arises from the "time advance" of the "nearby" geodesics, represented by the initial velocity of the Jacobi field in the u-direction. However the second case shows that this is not the only effect producing kinks and jumps and we will refer to it in the discussion of the next Section.

4 Discussion

We have shown that the geometry of impulsive pp-waves can be described entirely within the distributional picture. Note that it was essential to use regularization techniques, i.e. Colombeau's algebra, to handle the ill-defined singular terms instead of introducing "ad-hoc" multiplication rules into Schwartz linear theory. Even within the maximal class of regularizations of the Dirac-δ we were able to derive regularization independent distributional geodesics and deviation fields. This (mathematical) feature also has a remarkable physical consequence. Interpreting the impulsive wave as a limiting case of sandwich waves of the form (cf. [23]) $H(u, x, y) = f(x, y)\rho_\epsilon(u) \to f(x, y)\delta(u)$ we have shown that the impulsive limit is totally independent of the special form of the original profile ρ_ϵ (see also the results in [24]).

Finally (in the special case of plane waves) we discuss the relations of our approach to the one using the continuous form of the metric. The metric (2) has the advantage that simple particle motion can bee seen directly. Indeed free particles at fixed values of X, Y and $Z = V - T$ after the shock start to move such that their relative X- and Y-distance is given by the functions $1 + u_+$ and $1 - u_+$ respectively. This is in total agreement with equation (13). (Note that it is the coordinate transformation which introduces the motion in v-direction.)

Moreover, we can solve the geodesic equations for the metric (2) either by using the method of Sec. 3 or (since these equations only involve Heaviside and kink functions) by solving them separatly for $u < 0$ and $u > 0$ and joining them in a C^1-manner. Either way leads to the solutions ($u < 1$, and using analogous initial values as before)

$$
\begin{aligned}
X(u) &= x_0 + \dot{x}_0(2 + u_-) - \frac{\dot{x}_0}{1 + u_+} \\
Y(u) &= y_0 + \dot{y}_0 u_- + \frac{\dot{y}_0}{1 - u_+} \\
V(u) &= v_0 + \dot{v}_0(1 + u) + \frac{\dot{y}_0^2 u_+^2}{1 - u} - \frac{\dot{x}_0^2 u_+^2}{1 + u} \; ,
\end{aligned}
\tag{14}
$$

where $u_- := \theta(-u)u$.

If we now formally transform equations (14) according to (3) we again obtain the geodesics (11). Therefore we conclude that *physically* the two approaches to

impulsive plane waves, hence the two differential structures of the manifold, are equivalent. However, the transformation once more involves products of distributions ill-defined in the linear theory. Future work will be concerned with a mathematical analysis and interpretation of this discontinuous change of coordinates. Note that the transformation (3) is given precisely by the limit of the geodesic equations (11) with vanishing initial velocities. Hence we can study the respective transformation in the sandwich case and then approach the impulsive limit.

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